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Decomposable elements and ideals in semigroups

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1. Introduction. An element d [or an ideal D] of a semigroup S is called *decomposable* if there exist elements a, b [ideals A, B] in S such that $d=ab$ [$D=AB$]. In particular, an ideal D of S is called *idempotent* if $D^2=D$; it is said to be *left- [right-] reproduced* if $D=SD$ [$D=DS$] and it is said to be *reproduced* if $SD=D=DS$ ([3]). A semigroup in which every element is decomposable will be called a *semigroup with decomposable elements* and the analogous terminology will be used for the semigroups in which every ideal (or principal ideal) is decomposable or reproduced, and so on (cf. [5]).

Let $\mathcal{D}_e$ [$\mathcal{D}_p, \mathcal{D}_i$] denote the class of semigroups with decomposable elements [principal ideals, ideals]. Then $S\in\mathcal{D}_i$ implies $S\in\mathcal{D}_p$ and the latter implies $S\in\mathcal{D}_e$, obviously. Concerning the converse implications, our earlier investigations give, as direct consequences, the following results:

(i) *$S\in\mathcal{D}_e$ implies $S\in\mathcal{D}_p$ if S is commutative* ([4], Lemma 7); [1])

(ii) *$S\in\mathcal{D}_p$ implies $S\in\mathcal{D}_i$ if S is finite and commutative* ([6], Theorem 2).

It will be shown in Section 2 that neither (i) nor (ii) remains true if we omit (any one) of the conditions written there; consequently, $\mathcal{D}_e\supset\mathcal{D}_p\supset\mathcal{D}_i$.

An ideal A of a semigroup S is called *I-pure* ([2]) if

$$A\cap XS = XA \quad \text{and} \quad A\cap SX = AX \tag{1}$$

for any ideal X of S and it is said to be *weakly prime* if $XY\subseteqq A$ implies $X\subseteqq A$ or $Y\subseteqq A$ for each pair X, Y of ideals of S. Let $\mathcal{P}$ [$\mathcal{R}$, $\mathcal{I}$] denote the class of semigroups with I-pure [reproduced, idempotent] ideals. By Theorems 9—11 of [2], $\mathcal{P}\cap\mathcal{R}=\mathcal{I}$. In Section 3 we improve this result by showing $\mathcal{P}\cap\mathcal{D}_e=\mathcal{I}$. Finally, in Section 4 we prove that any weakly prime decomposable ideal is reproduced at least from one side.

For the notations and concepts not defined here, see [1].

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[1]) The analogous problem for prime ideals has been solved in [3], Satz 1.

2. On the classes $\mathscr{D}_e$, $\mathscr{D}_p$ and $\mathscr{D}_i$ of semigroups. The following example was constructed by András Botos (Szeged). Consider the semigroup S generated by the set $\{g_0, g_1, g_2, g_3\}$ and subject to the generating relations

$$g_0 = g_0^2, \quad g_1 = g_1 g_3, \quad g_2 = g_3 g_2, \quad g_3 = g_3^2.$$

S is obviously decomposable. Let A and B be any ideals of S such that $J(g_1 g_2) \subseteq AB$. Then $g_1 g_2 \in AB$ and the generating relations imply $g_1 \in A$, $g_2 \in B$. It follows that $g_1 g_0 g_2 \in AB$, too. But $g_1 g_0 g_2 \notin J(g_1 g_2)$ whence $J(g_1 g_2) \subset AB$. Thus we have got

Proposition 1. *There exist semigroups with decomposable elements that are not semigroups with decomposable principal ideals.*

It remains to solve the problem whether the class $\mathscr{D}_e \setminus \mathscr{D}_p$ contains also finite semigroups or not.

Let C denote the additive semigroup of all complex numbers $a+bi$ with $a \geqq 0$, $b \geqq 0$ and $a+b \neq 0$. Then every element and, consequently, every principal ideal of C is decomposable. The set

$$I = \{u+vi : u \geqq 1 \text{ or } v \geqq 1\}$$

is an ideal of C. Let A and B be ideals of C such that $I \subseteq A+B$. Then $1 \in A+B$. Since the number 1 can be decomposed in C only into the sum of two positive real numbers less than 1, there exists an $a_0 \in A$ with $a_0 < 1$. Similarly, $i \in A+B$ implies the existence of a $b_0 i \in B$ with $b_0 < 1$. It follows that $A+B$ contains an element $a_0 + b_0 i$ of C with $a_0, b_0 < 1$. Hence $I \subset A+B$ and we have got

Proposition 2. *There exist (infinite) commutative semigroups with decomposable principal ideals that are not semigroups with decomposable ideals.*

Consider, finally, the semigroup $F = \{0, a, b, c\}$ in which

$$bc = b, \quad ca = a, \quad cc = c \quad \text{and} \quad xy = 0 \quad \text{for any other pairs} \quad x, y \in F.$$

It is a semigroup with decomposable principal ideals:

$$J(0) = \{0\} = J(0) \cdot J(0), \qquad J(b) = \{0, b\} = J(b) \cdot F,$$

$$J(a) = \{0, a\} = F \cdot J\{a\}, \qquad J(c) = F = F^2.$$

The set $P = \{0, a, b\}$ is an ideal of F, too. Let A, B be any ideals of F such that $P \subseteq AB$. Then $a \in AB$ and $b \in AB$, implying $c \in A$ and $c \in B$, respectively. It follows, by $J(c) = F$, that $A = B = F$. Hence $P \subset AB$ and we have proved:

Proposition 3. *There exist (non-commutative) finite semigroups with decomposable principal ideals that are not semigroups with decomposable ideals.*

Remark. A semigroup N with 0 is called *nilpotent* if there exists a positive integer r such that $N^r=\{0\}$. Let S be a semigroup with decomposable elements. Then $S=S^2=S^3=\dots$. It follows that S cannot be nilpotent if $|S|>1$.

3. I-pure ideals in semigroups with decomposable elements. In order to improve the result

$$\mathscr{P}\cap\mathscr{R}=\mathscr{I}, \tag{2}$$

mentioned in the introduction, we begin with

Theorem 1. *Any I-pure ideal of a semigroup with decomposable elements is idempotent.*

Proof. Let A be an I-pure ideal of the semigroup. Applying the first equation in (1) for $X=S$ and the second one for $X=A$ we get $A\cap S^2=SA$ and $SA=A^2$, i.e.

$$A\cap S^2=A^2$$

(without making any restriction for S). If, in particular, $S^2=S$, then $A=A\cap S=$ $=A\cap S^2=A^2$. Thus the theorem is proved.

Remark. Zero semigroups Z with $|Z|>1$ furnish trivial examples for semigroups in which every ideal is I-pure but none of the elements except the 0 is decomposable.

Theorem 2. *The classes $\mathscr{P}$, $\mathscr{D}_e$ and $\mathscr{I}$ of semigroups satisfy the equation $\mathscr{P}\cap\mathscr{D}_e=\mathscr{I}$.*

Proof. Clearly, $\mathscr{P}\cap\mathscr{D}_e\subseteqq\mathscr{I}$ by Theorem 1. Thus, (2) implies $\mathscr{I}=\mathscr{P}\cap\mathscr{R}\subseteqq$ $\subseteqq\mathscr{P}\cap\mathscr{D}_i\subseteqq\mathscr{P}\cap\mathscr{D}_e\subseteqq\mathscr{I}$, i.e. $\mathscr{I}=\mathscr{P}\cap\mathscr{D}_e$, as asserted.

4. On decomposable ideals. In this section we prove

Theorem 3. *Let A be a decomposable ideal of a semigroup S. If A is weakly prime, too, then it is left- or right-reproduced.*

Proof. Let X, Y be ideals of S such that $A=XY$. Then $A\subseteqq X$ and $A\subseteqq Y$. If A is weakly prime, too, then at least one of the converse inclusions $X\subseteqq A$ and $Y\subseteqq A$ is true as well. In the first case $X=A$, whence we get

$$AS\subseteqq A=AY\subseteqq AS,$$

i.e. $A=AS$. Similarly, in the second case we get $A=SA$.

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